

Antiderivatives & u -Substitution

AP Calculus AB | Unit 6 | indefinite integrals, $+C$, substitution

1. Antiderivatives and $+C$

F is an antiderivative of f if $F' = f$. The indefinite integral is the whole family:

$$\int f(x) dx = F(x) + C.$$

x^2 , $x^2 + 1$ and $x^2 - 2$ all have derivative $2x$: a constant shifts the graph up or down without changing any slope. So $\int 2x dx = x^2 + C$.

2. The basic rules

$$\begin{aligned} \int x^n dx &= \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) & \int \frac{1}{x} dx &= \ln|x| + C \\ \int e^x dx &= e^x + C & \int \sec^2 x dx &= \tan x + C \\ \int \cos x dx &= \sin x + C & \int \sin x dx &= -\cos x + C \end{aligned}$$

Constants factor out and sums split, as with derivatives.

3. u -substitution: the chain rule, backwards

Four steps

1. Let u be the **inside** function.
2. Differentiate: find du . It must appear in the integral, up to a constant.
3. Rewrite everything in terms of u and integrate.
4. Substitute back: the answer is in terms of x .

$$\int \cos(x^2) 2x dx : \quad u = x^2, \quad du = 2x dx \quad \Rightarrow \quad \int \cos u du = \sin u + C = \sin(x^2) + C.$$

A missing constant can always be fixed:

$$\int x(x^2 + 1)^5 dx : \quad u = x^2 + 1, \quad x dx = \frac{1}{2} du \quad \Rightarrow \quad \frac{1}{2} \int u^5 du = \frac{(x^2 + 1)^6}{12} + C.$$

$$\int e^{3x} dx = \frac{e^{3x}}{3} + C \quad (u = 3x, \quad dx = \frac{1}{3} du).$$

4. Substitution in a definite integral

Change the limits

$$\int_0^2 2x(x^2 + 1)^3 dx : \quad u = x^2 + 1, \quad x = 0 \Rightarrow u = 1, \quad x = 2 \Rightarrow u = 5.$$

$$\int_1^5 u^3 du = \left[\frac{u^4}{4} \right]_1^5 = \frac{625 - 1}{4} = 156.$$

No need to go back to x . Never write $\int_0^2 u^3 du$: u with x -limits is false.

5. Choosing a technique, and checking

- **Expand:** $\int (x + 1)^2 dx = \int (x^2 + 2x + 1) dx = \frac{x^3}{3} + x^2 + x + C.$
- **Divide:** $\int \frac{x^2 + 1}{x} dx = \int \left(x + \frac{1}{x} \right) dx = \frac{x^2}{2} + \ln |x| + C.$
- **Find C** from a starting value: $f'(x) = 2x, f(1) = 5 \Rightarrow f(x) = x^2 + 4.$
- **Check** by differentiating your answer: you must get the integrand back.

Where students lose points

1. Forgetting $+C$ on an indefinite integral.
2. Not changing the limits after substituting.
3. Losing the constant: $x dx = \frac{1}{2} du$, not du .
4. $\int \sin x dx = \cos x$ (the sign).
5. The power rule on $\frac{1}{x}$ instead of $\ln |x|$.

Now practise it

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