Antiderivatives & u-Substitution

AP Calculus AB | Unit 6 | indefinite integrals, $+C$, substitution

1. Antiderivatives and $+C$

F is an antiderivative of f if $F' = f$. The indefinite integral is the whole family:

$$\int f(x)\,dx = F(x) + C.$$

x^2, x^2+1 and x^2-2 all have derivative $2x$: a constant shifts the graph up or down without changing any slope. So $\int 2x\,dx = x^2 + C$.

2. The basic rules

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

Constants factor out and sums split, as with derivatives.

3. u-substitution: the chain rule, backwards

Four steps

1. Let u be the **inside** function.
2. Differentiate: find du. It must appear in the integral, up to a constant.
3. Rewrite everything in terms of u and integrate.
4. Substitute back: the answer is in terms of x.

$$\int \cos(x^2)\,2x\,dx: \quad u = x^2,\ du = 2x\,dx \quad \Rightarrow \quad \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

A missing constant can always be fixed:

$$\int x(x^2+1)^5\,dx: \quad u = x^2+1,\ x\,dx = \tfrac{1}{2}\,du \quad \Rightarrow \quad \tfrac{1}{2}\int u^5\,du = \frac{(x^2+1)^6}{12} + C.$$

$$\int e^{3x}\,dx = \frac{e^{3x}}{3} + C \qquad (u = 3x,\ dx = \tfrac{1}{3}\,du).$$

 © 2026 S&D Academy

4. Substitution in a definite integral

Change the limits

$$\int_0^2 2x(x^2+1)^3\,dx: \qquad u=x^2+1, \quad x=0\Rightarrow u=1, \quad x=2\Rightarrow u=5.$$

$$\int_1^5 u^3\,du=\left[\frac{u^4}{4}\right]_1^5=\frac{625-1}{4}=156.$$

No need to go back to x. Never write $\int_0^2 u^3\,du$: u with x-limits is false.

5. Choosing a technique, and checking

- **Expand**: $\displaystyle\int (x+1)^2dx=\int (x^2+2x+1)\,dx=\frac{x^3}{3}+x^2+x+C.$
- **Divide**: $\displaystyle\int \frac{x^2+1}{x}\,dx=\int\left(x+\frac{1}{x}\right)dx=\frac{x^2}{2}+\ln|x|+C.$
- **Find** C from a starting value: $f'(x)=2x$, $f(1)=5\Rightarrow f(x)=x^2+4$.
- **Check** by differentiating your answer: you must get the integrand back.

6. Symmetry, arctan, and ln

Symmetry. If g is even, $\int_{-a}^{a} g=2\int_0^a g$ ($\int_0^4 g=9\Rightarrow\int_{-4}^4 g=18$). If h is odd, $\int_{-a}^0 h=-\int_0^a h$ and $\int_{-a}^a h=0$ ($\int_0^3 h=5\Rightarrow\int_{-3}^0 h=-5$).

Arctan. $\int\frac{du}{u^2+a^2}=\frac{1}{a}\arctan\frac{u}{a}+C$. Completing the square, $x^2+4x+13=(x+2)^2+9$, so $\int\frac{dx}{x^2+4x+13}=\frac{1}{3}\arctan\frac{x+2}{3}+C$.

ln from substitution. $\int\cot x\,dx=\int\frac{\cos x}{\sin x}dx=\ln|\sin x|+C$ ($u=\sin x$); $\int\frac{6x}{x^2+5}dx=3\ln(x^2+5)+C$ ($u=x^2+5$, no absolute value needed).

Where students lose points

1. Forgetting $+C$ on an indefinite integral.
2. Not changing the limits after substituting.
3. Losing the constant: $x\,dx=\frac{1}{2}\,du$, not du.
4. $\int\sin x\,dx=\cos x$ (the sign).
5. The power rule on $\frac{1}{x}$ instead of $\ln|x|$.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

sad-academy.com

AP® is a registered trademark of College Board, which is not affiliated with and does not endorse this material.

 © 2026 S&D Academy