

Continuity & the Intermediate Value Theorem

AP Calculus AB | Unit 1 | discontinuities, piecewise functions, IVT

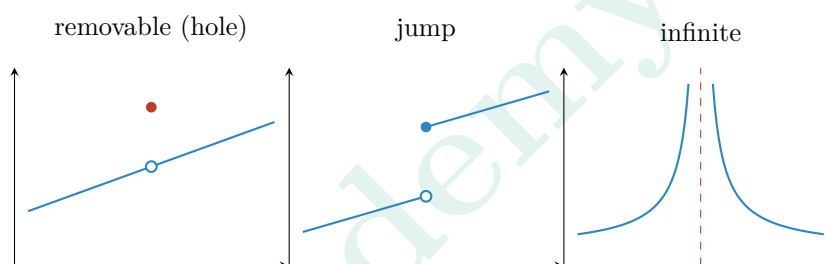
1. Continuity at a point

Informally: you can draw the graph without lifting your pencil. Precisely:

f is continuous at $x = a$ if all three hold

1. $f(a)$ is defined;
2. $\lim_{x \rightarrow a} f(x)$ exists (left and right limits agree);
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

2. Three kinds of discontinuity



- **Removable:** the limit exists but $f(a)$ is missing or different (condition 3 fails).
- **Jump:** left and right limits are different numbers (condition 2 fails).
- **Infinite:** $f(x) \rightarrow \pm\infty$ as $x \rightarrow a$ from at least one side — a **vertical asymptote** at $x = a$, e.g. $\frac{1}{x-2}$ at $x = 2$.

3. Fixing a discontinuity

Removable: define $f(a)$ to be the limit. $\frac{x^2 - 9}{x - 3}$ is undefined at 3, but $\lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = 6$, so setting $g(3) = 6$ makes g continuous. Cancelling alone does not remove the discontinuity; the value at 3 must be defined.

Piecewise: make the left limit, the right limit and the value agree. For $f(x) = x^2 + k$ ($x < 2$), $3x - 1$ ($x \geq 2$): $\lim_{x \rightarrow 2^-} f(x) = 4 + k$, $\lim_{x \rightarrow 2^+} f(x) = f(2) = 5$, so $4 + k = 5$ and $k = 1$.

Where to look: polynomials, sin, cos are continuous everywhere; rational functions everywhere except where the denominator is 0; roots, exponentials, logarithms on their domains. Discontinuities hide at zero denominators and at the seams of piecewise functions.

4. The Intermediate Value Theorem

IVT

If f is **continuous on** $[a, b]$ and N is between $f(a)$ and $f(b)$, then $f(c) = N$ for at least one c in (a, b) .

It guarantees that c *exists*; it does not say where c is or that there is only one. Without continuity there is no guarantee: a function with a jump can start below N , end above it, and never equal N .

With a table (typical FRQ). W is continuous, $W(2) = 7$, $W(5) = 15$. Must $W(t) = 10$ for some t in $(2, 5)$?

Yes. W is continuous on $[2, 5]$ and $W(2) = 7 < 10 < 15 = W(5)$, so by the Intermediate Value Theorem there is a time t in $(2, 5)$ with $W(t) = 10$.

Where students lose points

1. Using the IVT without stating that the function is continuous.
2. Checking only that the limit exists; continuity needs all three conditions.
3. Claiming the IVT tells you where c is, or that c is unique.
4. “Removing” a hole by cancelling, without defining the function at that point.
5. Mixing up the IVT (values of f) with the Mean Value Theorem (derivatives, Unit 5).

Remember

- Continuous at a : $f(a)$ defined, $\lim_{x \rightarrow a} f(x)$ exists, and they are equal.
- Removable (hole) • jump • infinite (vertical asymptote).
- Piecewise: left limit = right limit = $f(a)$.
- IVT: f continuous on $[a, b]$, N between $f(a)$ and $f(b) \Rightarrow f(c) = N$.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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