Continuity & the Intermediate Value Theorem

AP Calculus AB | Unit 1 | discontinuities, piecewise functions, IVT

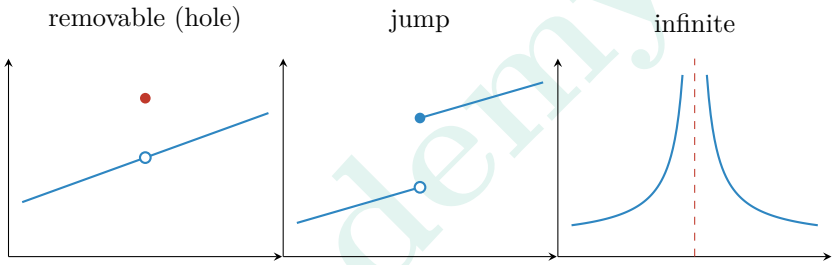
1. Continuity at a point

Informally: you can draw the graph without lifting your pencil. Precisely:

f is continuous at $x = a$ if all three hold

1. $f(a)$ is defined;
2. $\lim_{x \to a} f(x)$ exists (left and right limits agree);
3. $\lim_{x \to a} f(x) = f(a)$.

2. Three kinds of discontinuity



- **Removable**: the limit exists but $f(a)$ is missing or different (condition 3 fails).
- **Jump**: left and right limits are different numbers (condition 2 fails).
- **Infinite**: $f(x) \to \pm\infty$ as $x \to a$ from at least one side — a **vertical asymptote** at $x = a$, e.g. $\frac{1}{x-2}$ at $x = 2$.

3. Fixing a discontinuity

Removable: define $f(a)$ to be the limit. $\dfrac{x^2 - 9}{x - 3}$ is undefined at 3, but $\lim_{x \to 3} \dfrac{(x-3)(x+3)}{x-3} = 6$, so setting $g(3) = 6$ makes g continuous. Cancelling alone does not remove the discontinuity; the value at 3 must be defined.

Piecewise: make the left limit, the right limit and the value agree. For $f(x) = x^2 + k$ $(x < 2)$, $3x - 1$ $(x \geq 2)$: $\lim_{x \to 2^-} f(x) = 4 + k$, $\lim_{x \to 2^+} f(x) = f(2) = 5$, so $4 + k = 5$ and $k = 1$.

Where to look: polynomials, sin, cos are continuous everywhere; rational functions everywhere except where the denominator is 0; roots, exponentials, logarithms on their domains. Discontinuities hide at zero denominators and at the seams of piecewise functions.

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4. The Intermediate Value Theorem

IVT

If f is **continuous on** $[a, b]$ and N is between $f(a)$ and $f(b)$, then $f(c) = N$ for at least one c in (a, b).

It guarantees that c *exists*; it does not say where c is or that there is only one. Without continuity there is no guarantee: a function with a jump can start below N, end above it, and never equal N.

With a table (typical FRQ). W is continuous, $W(2) = 7$, $W(5) = 15$. Must $W(t) = 10$ for some t in $(2, 5)$?

Yes. W is continuous on $[2, 5]$ and $W(2) = 7 < 10 < 15 = W(5)$, so by the Intermediate Value Theorem there is a time t in $(2, 5)$ with $W(t) = 10$.

5. Holes, asymptotes, and counting with the IVT

Hole or asymptote? Factor the top and the bottom. A factor that *cancels* gives a hole (removable); a zero *left* in the denominator gives a vertical asymptote; a zero of the top alone is just an x-intercept. $\dfrac{x^2 - 5x - 14}{x^2 - 4x - 21} = \dfrac{(x-7)(x+2)}{(x-7)(x+3)}$: hole at $\left(7, \frac{9}{10}\right)$, vertical asymptote $x = -3$.

One value fixes only a hole. $f(x) = x + 5$ $(x < 2)$, $f(2) = k$, $f(x) = 10 - x^2$ $(x > 2)$: the one-sided limits are 7 and 6, a jump, so no k works. Famous limits: $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

Combinations. Sums, differences and products of continuous functions are continuous; quotients too, except where the bottom is 0. $g(f(x))$ is continuous at a if f is continuous at a and g is continuous at $f(a)$: if g jumps at $u = 2$, then $g(x - 6)$ is discontinuous at $x = 8$.

Counting with a table. Each sign change between neighbouring values gives at least one zero. For $f(x) = 6$, look at the signs of $f(x) - 6$. Values $5, -2, -4, 3, 7, -1$ at $x = 0, 1, 3, 4, 6, 9$: zeros in $(0, 1)$, $(3, 4)$, $(6, 9)$; $f = 6$ in $(4, 6)$ and $(6, 9)$. The IVT gives *at least* one; a **strictly** increasing or decreasing f has *exactly* one. Use an interval on which f is continuous: with a jump at $x = 4$, use $[5, 9]$, not $[2, 9]$.

Where students lose points

1. Using the IVT without stating that the function is continuous.
2. Checking only that the limit exists; continuity needs all three conditions.
3. Claiming the IVT tells you where c is, or that c is unique.
4. "Removing" a hole by cancelling, without defining the function at that point.
5. Mixing up the IVT (values of f) with the Mean Value Theorem (derivatives, Unit 5).

Remember

- Continuous at a: $f(a)$ defined, $\lim_{x\to a} f(x)$ exists, and they are equal.
- Removable (hole) • jump • infinite (vertical asymptote).
- Piecewise: left limit = right limit = $f(a)$.
- IVT: f continuous on $[a, b]$, N between $f(a)$ and $f(b) \Rightarrow f(c) = N$.

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Now practise it

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