

The Derivative as a Limit

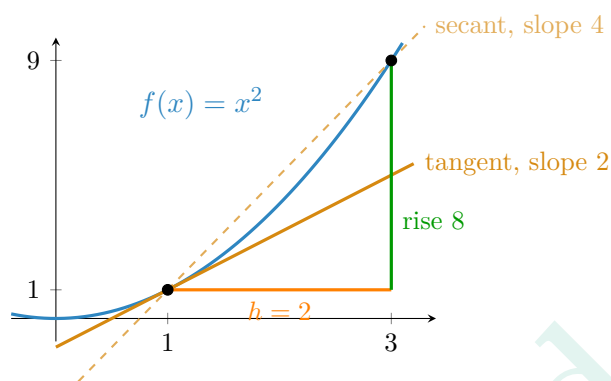
AP Calculus AB | CED topics 2.1–2.2 | plus differentiability (2.4)

1. Average vs. instantaneous

Drive 120 miles in 2 hours and your **average** speed is $120/2 = 60$ mph. But the speedometer shows something else: how fast you are going *right now*. That **instantaneous** rate of change is what a derivative measures. The question of this topic is how to get from the first to the second.

Secant slopes close in on the tangent slope

On $f(x) = x^2$, take the point $x = 1$ and a second point a step h to the right. The line through them is a **secant**; its slope (rise over run) is an average rate of change. Shrink h and watch the slope:



h	slope $\frac{f(1+h) - f(1)}{h}$
2	4
1	3
0.5	2.5
0.1	2.1
0.01	2.01
$\rightarrow 0$	$\rightarrow 2$

As $h \rightarrow 0$ the secant turns into the **tangent** line, and its slope, 2, is the derivative: $f'(1) = 2$.

Definition

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \text{or, equivalently,} \quad f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

The fraction is the **difference quotient**: rise $f(a+h) - f(a)$ over run h . f is **differentiable at a** when this limit exists (is a finite number).

Computing a derivative from the definition

Setting $h = 0$ straight away gives $\frac{0}{0}$, which says nothing; that is why we need a limit. Simplify first, cancel h , then let $h \rightarrow 0$:

$$f'(1) = \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \rightarrow 0} \frac{2h + h^2}{h} = \lim_{h \rightarrow 0} (2 + h) = 2.$$

Dividing by h is allowed because h approaches 0 but is never 0. The answer matches the table.

Reading a limit as a derivative

On the exam the limit usually comes *already written*, and the job is to recognise it. Match three things: the function f , the point a , and that the step in the numerator is *exactly* the denominator.

$$\lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} = f'(4) \text{ for } f(x) = \sqrt{x} \Rightarrow \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

When the derivative does not exist

Both one-sided limits of the difference quotient must exist and agree.

- **Corner:** one-sided slopes are finite but different. $|x - 2|$ at $x = 2$: -1 vs. $+1$.
- **Cusp:** one-sided slopes go to $+\infty$ and $-\infty$.
- **Vertical tangent:** the slope goes to $\pm\infty$ from both sides ($\sqrt[3]{x}$ at 0).
- **Discontinuity:** if f is not continuous at a , it is not differentiable there.

The one-way implication

differentiable at $a \implies$ continuous at a . The converse is false: $|x - 2|$ is continuous at 2 but not differentiable.

2. Two techniques the exam tests

When the step inside f is not the denominator

The definition needs the *same* quantity in the step and in the denominator. If they differ, rewrite until they match. With $f(5) = 2$ and $f'(5) = 4$:

$$\lim_{h \rightarrow 0} \frac{f(5 - 3h) - f(5)}{h} = \lim_{k \rightarrow 0} \frac{f(5 + k) - f(5)}{k} \cdot \frac{k}{h} = f'(5) \cdot (-3) = -12, \quad k = -3h.$$

The factor -3 is the ratio of the step to the denominator. Forgetting it gives 4; dividing by it instead gives $-\frac{4}{3}$.

Differentiability of a piecewise function

At the point where the pieces meet, check **two** things, in either order:

1. **Continuity:** the pieces give the same value there.
2. **Matching slopes:** the derivatives of the pieces give the same value there.

For $f(x) = 4x + 1$ on $x \leq 1$ and $2x^2 + b$ on $x > 1$: the slopes are 4 and $4x \rightarrow 4$, so they match for every b ; continuity needs $5 = 2 + b$, so $b = 3$. Slopes alone are not enough: with $b = 5$ both sides still have slope 4, but the graph jumps and $f'(1)$ does not exist.

Where students lose points

1. Reading off $f'(a)$ when the step inside f is not the denominator ($3h$ vs. h).
2. “ f is continuous, so it is differentiable.” Corners and cusps are continuous.
3. On a piecewise function, checking the slopes and forgetting continuity (or the reverse).
4. Recognising the derivative but naming the wrong f or the wrong a : in $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$ the point is $a = 4$, not 16.

Remember

- $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$: the limit of secant slopes, the slope of the tangent.
- Differentiable $\Rightarrow$ continuous, but not the other way round.
- Both one-sided limits of the difference quotient must exist and agree.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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