The Derivative as a Limit

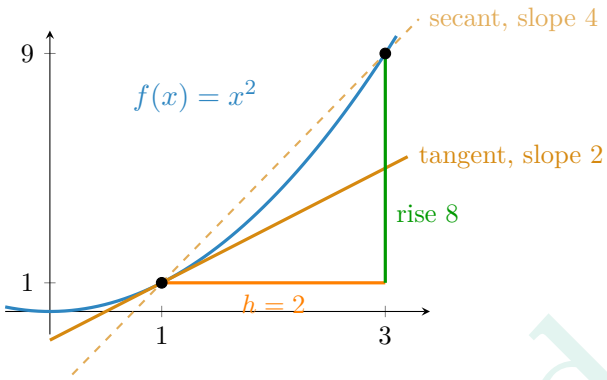
AP Calculus AB | CED topics 2.1–2.2 | plus differentiability (2.4)

1. Average vs. instantaneous

Drive 120 miles in 2 hours and your **average** speed is $120/2 = 60$ mph. But the speedometer shows something else: how fast you are going *right now*. That **instantaneous** rate of change is what a derivative measures. The question of this topic is how to get from the first to the second.

Secant slopes close in on the tangent slope

On $f(x) = x^2$, take the point $x = 1$ and a second point a step h to the right. The line through them is a **secant**; its slope (rise over run) is an average rate of change. Shrink h and watch the slope:



h	slope $\dfrac{f(1+h)-f(1)}{h}$
2	4
1	3
0.5	2.5
0.1	2.1
0.01	2.01
$\to 0$	$\to$ **2**

As $h \to 0$ the secant turns into the **tangent** line, and its slope, 2, is the derivative: $f'(1) = 2$.

Definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad \text{or, equivalently,} \qquad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

The fraction is the **difference quotient**: rise $f(a+h)-f(a)$ over run h. f is **differentiable at** a when this limit exists (is a finite number).

Computing a derivative from the definition

Setting $h = 0$ straight away gives $\frac{0}{0}$, which says nothing; that is why we need a limit. Simplify first, cancel h, then let $h \to 0$:

$$f'(1) = \lim_{h\to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0} (2+h) = 2.$$

Dividing by h is allowed because h approaches 0 but is never 0. The answer matches the table.

Reading a limit as a derivative

On the exam the limit usually comes *already written*, and the job is to recognise it. Match three things: the function f, the point a, and that the step in the numerator is *exactly* the denominator.

$$\lim_{h\to 0} \frac{\ln(2+h) - \ln 2}{h} = f'(2) \text{ for } f(x) = \ln x \;\Rightarrow\; \frac{1}{2}, \qquad \lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h} = f'(9) \text{ for } f(x) = \sqrt{x} \;\Rightarrow\; \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

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In the other form, $\lim_{x\to 2}\dfrac{x^3-8}{x-2} = f'(2)$ for $f(x)=x^3$, which is $3(2)^2=12$; and $\lim_{h\to 0}\dfrac{\sin(\frac{\pi}{3}+h)-\frac{\sqrt{3}}{2}}{h} = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$.

With a fraction. $f(x)=\frac{4}{x}$: $f'(2)=\lim_{h\to 0}\dfrac{\frac{4}{2+h}-2}{h}=\lim_{h\to 0}\dfrac{-2h}{h(2+h)}=-\dfrac{1}{2}$. **At a seam**, use the definition itself: $x|x|$ at 0 gives $\lim|h|=0$ (differentiable); $x\sin\frac{1}{x}$ with value 0 at 0 gives $\lim\sin\frac{1}{h}$, which does not exist.

When the derivative does not exist

Both one-sided limits of the difference quotient must exist and agree.

- **Corner**: one-sided slopes are finite but different. $|x-2|$ at $x=2$: -1 vs. $+1$.
- **Cusp**: one-sided slopes go to $+\infty$ and $-\infty$.
- **Vertical tangent**: the slope goes to $\pm\infty$ from both sides ($\sqrt[3]{x}$ at 0).
- **Discontinuity**: if f is not continuous at a, it is not differentiable there.

The one-way implication

differentiable at $a \implies$ continuous at a. The converse is false: $|x-2|$ is continuous at 2 but not differentiable.

2. Two techniques the exam tests

When the step inside f is not the denominator

The definition needs the *same* quantity in the step and in the denominator. If they differ, rewrite until they match. With $f(5)=2$ and $f'(5)=4$:

$$\lim_{h\to 0}\frac{f(5-3h)-f(5)}{h}=\lim_{k\to 0}\frac{f(5+k)-f(5)}{k}\cdot\frac{k}{h}=f'(5)\cdot(-3)=-12, \qquad k=-3h.$$

The factor -3 is the ratio of the step to the denominator. Forgetting it gives 4; dividing by it instead gives $-\frac{4}{3}$.

Differentiability of a piecewise function

At the point where the pieces meet, check **two** things, in either order:

1. **Continuity**: the pieces give the same value there.
2. **Matching slopes**: the derivatives of the pieces give the same value there.

For $f(x)=x^2-2$ on $x\le 3$ and $ax+b$ on $x>3$: the slopes are $2x=6$ and a, so $a=6$; continuity needs $7=3a+b$, so $b=-11$. Slopes alone are not enough: if the values do not match, the graph jumps and $f'(3)$ does not exist.

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Where students lose points

1. Reading off $f'(a)$ when the step inside f is not the denominator ($3h$ vs. h).
2. "f is continuous, so it is differentiable." Corners and cusps are continuous.
3. On a piecewise function, checking the slopes and forgetting continuity (or the reverse).
4. Recognising the derivative but naming the wrong f or the wrong a: in $\lim_{x\to 4} \frac{x^2-16}{x-4}$ the point is $a = 4$, not 16.

Remember

- $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$: the limit of secant slopes, the slope of the tangent.
- Differentiable $\Rightarrow$ continuous, but not the other way round.
- Both one-sided limits of the difference quotient must exist and agree.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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