

Differential Equations

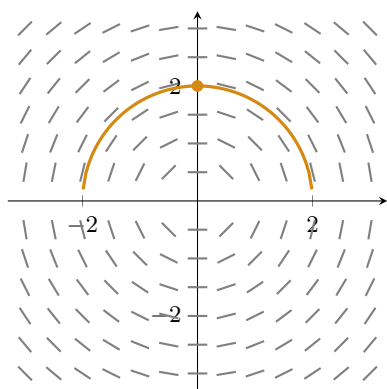
AP Calculus AB | Unit 7 | slope fields, separation of variables, exponential models

1. What a differential equation is

An equation that involves an unknown function *and* its derivative. Its solution is a function, not a number.

Verifying a solution. Plug it into both sides. Is $y = e^{2x}$ a solution of $\frac{dy}{dx} = 2y$? Left side: $\frac{dy}{dx} = 2e^{2x}$. Right side: $2y = 2e^{2x}$. Equal for every x , so yes.

2. Slope fields



For $\frac{dy}{dx} = -\frac{x}{y}$: at each point the equation gives a slope $((1, 1) \rightarrow -1, (2, 1) \rightarrow -2, (0, 2) \rightarrow 0)$. Draw a short segment with that slope at every grid point: a **slope field**. A solution is a curve that follows the segments; through $(0, 2)$ it is the top half of a circle. **Matching a field to an equation:** where are the slopes zero? Do they change with x only, with y only, or with both?

3. Separation of variables

Four steps: $\frac{dy}{dx} = -\frac{x}{y}, y(0) = 2$

1. **Separate:** $y dy = -x dx$.

2. **Integrate** both sides, with $+C$: $\frac{y^2}{2} = -\frac{x^2}{2} + C$.

3. Use the **initial condition**: $\frac{2^2}{2} = 0 + C \Rightarrow C = 2$.

4. **Solve for y :** $y^2 = 4 - x^2 \Rightarrow y = \sqrt{4 - x^2}$ (positive root, because $y(0) = 2 > 0$).

How it is scored. No separation of variables: no points for the part. No $+C$: at most the first two points. Then come the antiderivatives, the use of the initial condition, and solving for y .

4. The exponential model

$$\frac{dy}{dt} = ky \implies \frac{1}{y} dy = k dt \implies \ln |y| = kt + C \implies y = y_0 e^{kt}.$$

A rate proportional to the amount means exponential growth ($k > 0$) or decay ($k < 0$).

Example. $\frac{dy}{dt} = 0.5y, y(0) = 100$ gives $y = 100e^{0.5t}$. Doubling time: $e^{0.5t} = 2 \Rightarrow t = \frac{\ln 2}{0.5} \approx 1.39$.

$$\frac{dy}{dt} = -0.2y, y(0) = 50 \text{ gives } y = 50e^{-0.2t}.$$

Where students lose points

1. Not separating the variables.
2. Forgetting $+C$, or adding it after solving for y .
3. Taking the wrong sign of the square root.
4. Writing e^{kt+C} as $e^{kt} + C$ (it is a constant *times* e^{kt}).
5. Slope field: zero or undefined slopes in the wrong place.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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