Volumes of Revolution

AP Calculus AB | Unit 8 | the disc and washer methods

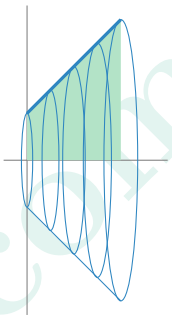
Spin a region around a line and it sweeps out a solid. Every slice perpendicular to the axis is a circle (a **disc**) or a ring (a **washer**).

1. Discs

When the region touches the axis, each slice is a disc of radius $R(x)$, with area $\pi R(x)^2$.

Disc method

$$V = \pi \int_a^b R(x)^2 \, dx.$$



Example. Under $y = x + 1$ on $[0, 2]$, about the x-axis: $V = \pi \int_0^2 (x+1)^2 \, dx = \pi \left[\frac{(x+1)^3}{3}\right]_0^2 = \frac{26\pi}{3}$ (a cone with its tip cut off).

2. Washers

When the region does not touch the axis, each slice is a washer: an outer disc of radius R with an inner disc of radius r removed.

Washer method

$$V = \pi \int_a^b (R^2 - r^2) \, dx \qquad \text{(square each radius, then subtract).}$$

Example. Between $y = x$ and $y = \frac{x^2}{3}$ (they meet at $x = 0$ and $x = 3$), about the x-axis: $R = x$, $r = \frac{x^2}{3}$, so $V = \pi \int_0^3 \left(x^2 - \frac{x^4}{9}\right) dx = \frac{18\pi}{5}$. Trap: $\pi \int_0^3 (R - r)^2 dx = \frac{9\pi}{10}$, a quarter of the right answer.

3. Other axes: measure every radius from the axis

- **About** $y = -1$ (below the region): each radius gets one unit longer. $R = x + 1$, $r = \frac{x^2}{3} + 1$, and $V = \pi \int_0^3 \left[(x+1)^2 - \left(\frac{x^2}{3} + 1\right)^2\right] dx = \frac{33\pi}{5}$.
- **About** $y = 3$ (above the region): radius = axis − curve, and the *bottom* curve is now farther away. $R = 3 - \frac{x^2}{3}$, $r = 3 - x$, and $V = \pi \int_0^3 \left[\left(3 - \frac{x^2}{3}\right)^2 - (3 - x)^2\right] dx = \frac{27\pi}{5}$.

4. About the y-axis

A vertical axis means horizontal slices: write each radius as x in terms of y and integrate with respect to y. For the region bounded by $y = x^2$, the y-axis and $y = 3$, the radius is $x = \sqrt{y}$:

$$V = \pi \int_0^3 (\sqrt{y})^2 dy = \pi \int_0^3 y \, dy = \frac{9\pi}{2}.$$

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Where students lose points

1. $(R-r)^2$ instead of R^2-r^2.
2. Forgetting π.
3. Radii not measured from the axis of rotation (for example, using the curves themselves about $y=-1$).
4. Rotating about the y-axis but writing the radius and the limits in x.
5. Swapping outer and inner radii when the axis is above the region.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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