Second and Higher-Order Derivatives

AP Calculus AB | Unit 3 | the derivative of the derivative

1. Differentiate again

A derivative is a function, so it can be differentiated again.

$$f(x) = x^4, \quad f'(x) = 4x^3, \quad f''(x) = 12x^2, \quad f'''(x) = 24x, \quad f^{(4)}(x) = 24, \quad f^{(5)}(x) = 0.$$

Notation and meaning

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right), \qquad f^{(n)}(x) = \text{the } n\text{th derivative.}$$

The second derivative is the rate of change of the rate of change: position → velocity → acceleration, each by one derivative.

2. Patterns

- The derivatives of $\sin x$ cycle every four steps: $\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x$. Only the remainder after dividing by 4 matters: $99 = 4 \cdot 24 + 3$, so $\dfrac{d^{99}}{dx^{99}} \sin x = -\cos x$.
- Each derivative of e^{kx} brings out one factor of k: $\dfrac{d^n}{dx^n} e^{2x} = 2^n e^{2x}$.
- A polynomial of degree n has $(n+1)$st derivative 0.

3. Rules inside rules

The second step is often a product. For $y = e^{x^2}$, the chain rule gives $y' = 2x\, e^{x^2}$, which is a product, so

$$y'' = 2 \cdot e^{x^2} + 2x \cdot e^{x^2} \cdot 2x = \left(4x^2 + 2\right) e^{x^2}, \qquad y''(0) = 2.$$

From a table. For $h(x) = f(3x)$: $h'(x) = 3f'(3x)$ and $h''(x) = 9f''(3x)$. If $f''(6) = -2$, then $h''(2) = 9f''(6) = -18$: the table is read at 6, not at 2.

4. The second derivative of an implicit curve

On $x^2 + y^2 = 25$, implicit differentiation gives $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Then, with the quotient rule,

$$\frac{d^2y}{dx^2} = -\frac{(1)(y) - x\,\frac{dy}{dx}}{y^2} = -\frac{y + x \cdot \frac{x}{y}}{y^2} = -\frac{x^2 + y^2}{y^3} = -\frac{25}{y^3}.$$

1. Differentiate $\frac{dy}{dx}$; a $\frac{dy}{dx}$ appears inside.
2. Substitute the known $\frac{dy}{dx}$, so only x and y remain.
3. Simplify with the curve's own equation ($x^2 + y^2 = 25$).

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At $(3, 4)$: $\frac{d^2y}{dx^2} = -\frac{25}{64} < 0$, so the top of the circle is concave down.

Where students lose points

1. Squaring f' instead of differentiating it again.
2. Missing that the first derivative is a product, so the second step needs the product rule.
3. Losing a chain-rule factor: $f(3x)$ has second derivative $9f''(3x)$.
4. Implicit: leaving $\frac{dy}{dx}$ in the second derivative instead of substituting it.
5. Cycles: miscounting the remainder after dividing by the length of the cycle.

Now practise it

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