

Implicit & Inverse Differentiation

AP Calculus AB | Unit 3 | implicit curves, inverse functions, higher-order derivatives

1. Implicit differentiation

A curve such as $x^2 + y^2 = 25$ is not a function $y = f(x)$, but near a point y still *behaves* like a function of x . So every y is an “inside function” and the chain rule applies.

The one idea

$$\frac{d}{dx} [x^2] = 2x \quad \text{but} \quad \frac{d}{dx} [y^2] = 2y \frac{dy}{dx}$$

Every time you differentiate a y , attach $\frac{dy}{dx}$.

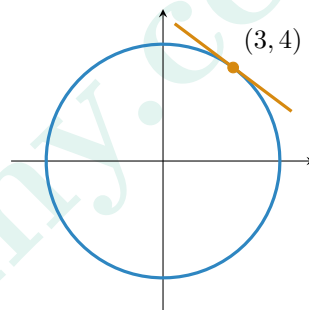
Three steps for $x^2 + y^2 = 25$ at $(3, 4)$:

1. Differentiate both sides: $2x + 2y \frac{dy}{dx} = 0$.
2. Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{x}{y}$.
3. Plug in the point: $-\frac{3}{4}$.

The answer usually contains both x and y .

When x and y are multiplied, use the product rule. For $x^2 + xy + y^2 = 7$:

$$2x + \left(y + x \frac{dy}{dx}\right) + 2y \frac{dy}{dx} = 0 \implies (x + 2y) \frac{dy}{dx} = -(2x + y) \implies \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$



2. The derivative of an inverse function

Reflecting over $y = x$ swaps x and y , so rise and run swap: the slopes are reciprocals.

Inverse function rule

$$(f^{-1})'(b) = \frac{1}{f'(a)} \quad \text{where } f(a) = b.$$

With a table. Given $f(1) = 2$, $f'(1) = 4$, $f(2) = 10$, $f'(2) = 13$ and $g = f^{-1}$: first find where f outputs 2. $f(1) = 2$, so $g(2) = 1$ and $g'(2) = \frac{1}{f'(1)} = \frac{1}{4}$. The trap answer is $\frac{1}{f'(2)} = \frac{1}{13}$.

3. arcsin, arccos, arctan

If $y = \arcsin x$ then $\sin y = x$, so $\cos y \frac{dy}{dx} = 1$ and $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$.

$$\frac{d}{dx} [\arcsin x] = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} [\arccos x] = -\frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} [\arctan x] = \frac{1}{1+x^2}.$$

The chain rule still applies: $\frac{d}{dx} [\arctan(3x)] = \frac{1}{1+(3x)^2} \cdot 3 = \frac{3}{1+9x^2}$.

4. Higher-order derivatives

Differentiate again: $f(x) = x^4 \Rightarrow f'(x) = 4x^3 \Rightarrow f''(x) = 12x^2 \Rightarrow f'''(x) = 24x$. Notation: $f''(x) = \frac{d^2y}{dx^2}$, the rate of change of the rate of change (position $\rightarrow$ velocity $\rightarrow$ acceleration).

Second derivative of an implicit curve: substitute dy/dx

For the circle, $\frac{dy}{dx} = -\frac{x}{y}$. Quotient rule, then replace $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = -\frac{(1)(y) - (x)\frac{dy}{dx}}{y^2} = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{x^2 + y^2}{y^3} = -\frac{25}{y^3}.$$

Where students lose points

1. Forgetting $\frac{dy}{dx}$ when differentiating a y term.
2. Treating xy as one term instead of a product.
3. Inverse: using $f'(b)$ instead of $f'(a)$, where $f(a) = b$.
4. Implicit second derivative: not substituting $\frac{dy}{dx}$ back in.
5. $\arctan(3x)$: forgetting the chain rule, or writing $1 + 3x^2$.

Now practise it

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