Implicit Differentiation

AP Calculus AB | Unit 3 | $\frac{dy}{dx}$ without solving for y

1. Why every y gets a $\frac{dy}{dx}$

Along a curve such as $x^2 + y^2 = 25$, y is a function of x even when we cannot write it as one formula. So a term like y^2 is a function inside a function, and the chain rule applies.

Differentiate a y-term as usual, then multiply by $\frac{dy}{dx}$

$$\frac{d}{dx}\left[y^2\right] = 2y\,\frac{dy}{dx}, \qquad \frac{d}{dx}\left[y^3\right] = 3y^2\,\frac{dy}{dx}, \qquad \frac{d}{dx}\left[e^y\right] = e^y\,\frac{dy}{dx},$$

$$\frac{d}{dx}\left[\sin y\right] = \cos y\,\frac{dy}{dx}, \qquad \frac{d}{dx}\left[\ln y\right] = \frac{1}{y}\,\frac{dy}{dx}.$$

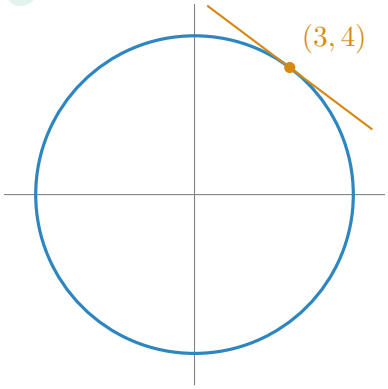
An x-term needs no extra factor: $\frac{d}{dx}\left[x^2\right] = 2x$.

2. Three steps

Example. The slope of $x^2 + y^2 = 25$ at $(3, 4)$.

1. Differentiate both sides: $2x + 2y\,\frac{dy}{dx} = 0$.
2. Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{x}{y}$.
3. Plug in the point: $\frac{dy}{dx} = -\frac{3}{4}$.

Tangent line: $y - 4 = -\frac{3}{4}(x - 3)$. The slope of an implicit curve usually depends on both x and y.



3. When x and y are multiplied

xy is a product, so it needs the product rule: $\frac{d}{dx}[xy] = 1 \cdot y + x\,\frac{dy}{dx}$.

Example. $x^2 + xy + y^2 = 3$.

$$2x + \left(y + x\,\frac{dy}{dx}\right) + 2y\,\frac{dy}{dx} = 0 \implies (x + 2y)\,\frac{dy}{dx} = -(2x + y) \implies \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

Collect every $\frac{dy}{dx}$ term on one side, factor, then divide. The point $(1, 1)$ is on the curve $(1 + 1 + 1 = 3)$, the slope there is $-\frac{2+1}{1+2} = -1$, and the tangent line is $y = 2 - x$.

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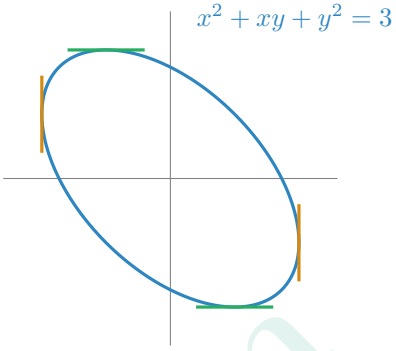
4. Horizontal and vertical tangents

With $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$:

- **Horizontal**: the top is 0. $2x + y = 0$, so $y = -2x$. Into the curve: $x^2 - 2x^2 + 4x^2 = 3x^2 = 3$, $x = \pm 1$. Points $(1, -2)$ and $(-1, 2)$.
- **Vertical**: the bottom is 0. $x + 2y = 0$; the same substitution gives $(-2, 1)$ and $(2, -1)$.
- If the top and the bottom are both 0 at a point, this test says nothing there.



Setting the top or the bottom to 0 gives a *line*; the points come from substituting back into the curve.

Where students lose points

1. Forgetting $\frac{dy}{dx}$ when differentiating a y-term.
2. Treating xy as one term instead of a product.
3. Dividing before every $\frac{dy}{dx}$ term has been collected and factored out.
4. Horizontal or vertical tangents: setting the top or bottom to 0 and not substituting back into the curve.
5. Plugging in a point that is not on the curve; check it in the equation first.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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