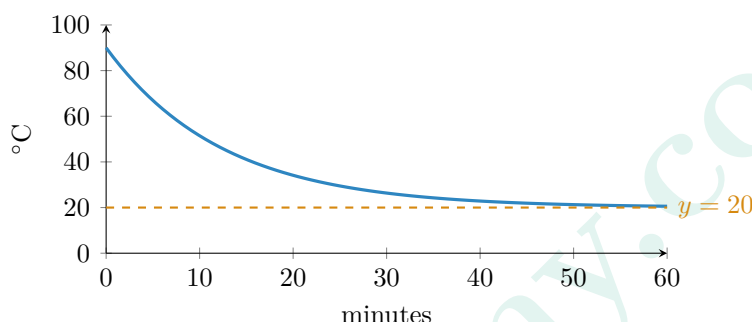


Limits at Infinity & Horizontal Asymptotes

AP Calculus AB | Unit 1 | end behavior, the degree rule, growth rates

1. What $x \rightarrow \infty$ means

$\lim_{x \rightarrow \infty} f(x) = L$ means $f(x)$ gets as close to L as we like once x is large enough. Then the line $y = L$ is a **horizontal asymptote**; check $x \rightarrow -\infty$ too. A coffee at 90°C in a 20°C room follows $T(t) = 20 + 70e^{-0.08t}$: $\lim_{t \rightarrow \infty} T(t) = 20$, the room temperature, which the model never reaches.



2. Rational functions

Divide every term by the highest power of x in the denominator:

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 1}{x^2 + 5} = \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x^2}}{1 + \frac{5}{x^2}} = \frac{3 + 0}{1 + 0} = 3.$$

The degree rule

- degree of top < degree of bottom $\Rightarrow$ limit 0 ($\frac{2x+1}{x^2+1} \rightarrow 0$);
- degrees equal $\Rightarrow$ ratio of leading coefficients ($\frac{5x^3-x}{2x^3+7} \rightarrow \frac{5}{2}$);
- degree of top > degree of bottom $\Rightarrow$ no horizontal asymptote ($\rightarrow \pm\infty$).

Read the leading coefficients carefully: $\frac{1-2x^2}{x^2+3} \rightarrow -2$.

3. Who grows fastest

In the long run $\ln x \ll x^n \ll e^x$: exponentials beat every power, powers beat logarithms. The faster function on the bottom sends the quotient to 0:

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = 0, \quad \lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0.$$

4. Two traps

Square roots at $-\infty$. $\sqrt{x^2} = |x|$, and for $x < 0$, $|x| = -x$. So

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{x + 3} = 2, \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{x + 3} = -2 :$$

two different horizontal asymptotes. Check $x \rightarrow -\infty$ separately whenever there is a root.

Crossing is allowed. A graph can cross its horizontal asymptote ($\frac{\sin x}{x} \rightarrow 0$ crosses $y = 0$ again and again). A horizontal asymptote describes the far ends only.

Horizontal vs. vertical asymptotes

Horizontal: $\lim_{x \rightarrow \pm\infty} f(x) = L$ (x goes to infinity, f approaches a number).

Vertical: $\lim_{x \rightarrow a} f(x) = \pm\infty$ (x approaches a number, f goes to infinity).

Where students lose points

1. Mixing up horizontal and vertical asymptotes.
2. Forgetting to check $x \rightarrow -\infty$ separately with square roots.
3. Using the wrong leading coefficients (signs!).
4. Believing a graph cannot cross its horizontal asymptote.
5. Getting the growth race backwards.

Remember

- $\lim_{x \rightarrow \pm\infty} f(x) = L \Rightarrow y = L$ is a horizontal asymptote.
- Rational: bottom bigger $\rightarrow 0$; equal $\rightarrow$ leading coefficients; top bigger $\rightarrow$ no HA.
- $\ln x \ll x^n \ll e^x$.
- $\sqrt{x^2} = |x|$: check $x \rightarrow -\infty$ separately.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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