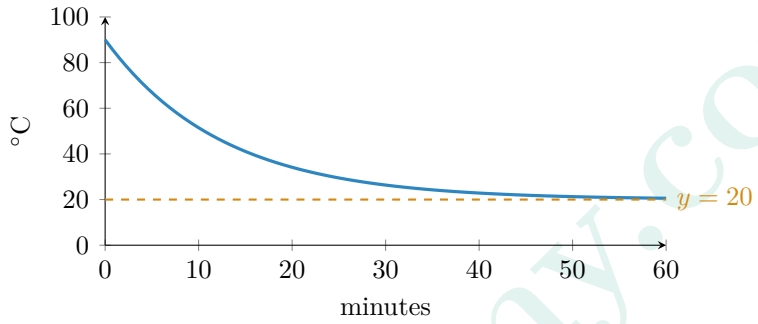
Limits at Infinity & Horizontal Asymptotes

AP Calculus AB | Unit 1 | end behavior, the degree rule, growth rates

1. What $x \to \infty$ means

$\lim_{x\to\infty} f(x) = L$ means $f(x)$ gets as close to L as we like once x is large enough. Then the line $y = L$ is a **horizontal asymptote**; check $x \to -\infty$ too. A coffee at 90°C in a 20°C room follows $T(t) = 20 + 70e^{-0.08t}$: $\lim_{t\to\infty} T(t) = 20$, the room temperature, which the model never reaches.



2. Rational functions

Divide every term by the highest power of x in the denominator:

$$\lim_{x\to\infty} \frac{3x^2+1}{x^2+5} = \lim_{x\to\infty} \frac{3+\frac{1}{x^2}}{1+\frac{5}{x^2}} = \frac{3+0}{1+0} = 3.$$

The degree rule

- degree of top < degree of bottom ⇒ limit 0 $\left(\frac{2x+1}{x^2+1} \to 0\right)$;
- degrees equal ⇒ ratio of leading coefficients $\left(\frac{5x^3-x}{2x^3+7} \to \frac{5}{2}\right)$;
- degree of top > degree of bottom ⇒ no horizontal asymptote ($\to \pm\infty$).

Read the leading coefficients carefully: $\frac{1-2x^2}{x^2+3} \to -2$.

3. Who grows fastest

In the long run $\ln x \ll x^n \ll e^x$: exponentials beat every power, powers beat logarithms. The faster function on the bottom sends the quotient to 0:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0, \qquad \lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

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4. Two traps

Square roots at $-\infty$. $\sqrt{x^2} = |x|$, and for $x < 0$, $|x| = -x$. So

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{x+3} = 2, \qquad \lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{x+3} = -2 :$$

two different horizontal asymptotes. Check $x \to -\infty$ separately whenever there is a root.

Crossing is allowed. A graph can cross its horizontal asymptote ($\frac{\sin x}{x} \to 0$ crosses $y = 0$ again and again). A horizontal asymptote describes the far ends only.

Horizontal vs. vertical asymptotes

Horizontal: $\lim_{x\to\pm\infty} f(x) = L$ (x goes to infinity, f approaches a number).
Vertical: $\lim_{x\to a} f(x) = \pm\infty$ (x approaches a number, f goes to infinity).

5. Exponentials at both ends

As $x \to \infty$, 2^x, 3^x and e^x grow without bound; as $x \to -\infty$ they all go to 0. So $\lim_{x\to-\infty} \frac{5+2^x}{4-3^x} = \frac{5}{4}$. A function built from e^x can have two different horizontal asymptotes: for $f(x) = \frac{4e^x-1}{e^x+2}$, dividing by e^x gives $\lim_{x\to\infty} f(x) = 4$, while $\lim_{x\to-\infty} f(x) = \frac{0-1}{0+2} = -\frac{1}{2}$; check both ends.

Horizontal and vertical asymptotes are separate checks: horizontal from $x \to \pm\infty$, vertical where the denominator is 0 and the numerator is not. $\frac{3x^2+1}{x^2+2}$ has $y = 3$ and no vertical asymptote; $\frac{3x^2+1}{x^2-4}$ has $y = 3$ and $x = \pm 2$; $\frac{4e^x-1}{e^x+2}$ has no vertical asymptote, because $e^x + 2$ is never 0.

Where students lose points

1. Mixing up horizontal and vertical asymptotes.
2. Forgetting to check $x \to -\infty$ separately with square roots.
3. Using the wrong leading coefficients (signs!).
4. Believing a graph cannot cross its horizontal asymptote.
5. Getting the growth race backwards.

Remember

- $\lim_{x\to\pm\infty} f(x) = L \Rightarrow y = L$ is a horizontal asymptote.
- Rational: bottom bigger $\to 0$; equal $\to$ leading coefficients; top bigger $\to$ no HA.
- $\ln x \ll x^n \ll e^x$.
- $\sqrt{x^2} = |x|$: check $x \to -\infty$ separately.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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