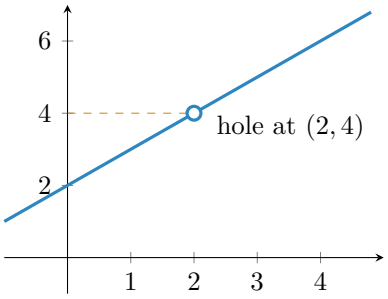
Limits: What They Mean & How to Find Them

AP Calculus AB | Unit 1 | limit notation, one-sided limits, algebraic techniques

1. What a limit means

A limit asks: *as x gets closer and closer to a, what value does $f(x)$ get closer to?* Take $f(x) = \dfrac{x^2-4}{x-2}$. At $x = 2$ it gives $\frac{0}{0}$, but near $x = 2$ the values close in on 4 from both sides:



x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

Notation

$$\lim_{x\to 2} f(x) = 4 \qquad \text{“the limit of } f(x) \text{ as } x \text{ approaches 2 is 4”}$$

The limit is about where f is **heading**, not what $f(2)$ is. If someone defines $f(2) = 1$, the limit is still 4; $f(2)$ can be anything, or not exist at all.

2. One-sided limits

$\lim\limits_{x\to a^-} f(x)$ is the value approached from the **left**, $\lim\limits_{x\to a^+} f(x)$ from the **right**. For a jump where the left side heads to 1 and the right side to 3, the two disagree, so $\lim\limits_{x\to a} f(x)$ **does not exist**.

When does a limit exist?

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

3. Direct substitution and limit laws

If f has no break at a (polynomials; rational functions with a nonzero denominator; roots, exponentials, sin, cos), just plug in: $\lim\limits_{x\to 3}(x^2+1) = 3^2+1 = 10$. This rests on the limit laws:

$$\lim(f \pm g) = \lim f \pm \lim g, \qquad \lim(f \cdot g) = \lim f \cdot \lim g, \qquad \lim \frac{f}{g} = \frac{\lim f}{\lim g} \quad (\lim g \neq 0).$$

4. When substitution gives $\frac{0}{0}$

$\frac{0}{0}$ is not an answer; it means *simplify first.* Keep writing “lim” on every line until you substitute.

© 2026 S&D Academy

Factor and cancel

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4.$$

Cancelling is allowed because $x\to 2$ but $x\neq 2$, so $x-2\neq 0$. The two functions are the same line except for the one missing point, and the limit ignores that point.

Multiply by the conjugate

$$\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}=\lim_{x\to 0}\frac{(x+9)-9}{x\,(\sqrt{x+9}+3)}=\lim_{x\to 0}\frac{1}{\sqrt{x+9}+3}=\frac{1}{6}.$$

A limit to know by heart

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad (x \text{ in radians; proved with the squeeze theorem}).$$

5. More one-sided limits, and more tools

Absolute values. Pick a side and fix the sign: for $x<4$, $|x-4|=-(x-4)$, so $\lim_{x\to 4^-}\frac{5|x-4|}{x-4}=-5$, while $\lim_{x\to 4^+}=5$; the two-sided limit does not exist. Likewise $\lim_{x\to 3^+}\frac{x^2-9}{|x-3|}=6$ and $\lim_{x\to 3^-}\frac{x^2-9}{|x-3|}=-6$.

Pieces. Each one-sided limit uses the piece on that side: $f(x)=2x+3$ $(x<2)$, x^2+1 $(x\geq 2)$ gives 7 from the left and 5 from the right, so $\lim_{x\to 2}f(x)$ does not exist. **Infinite limits:** $\lim_{x\to 3^-}\frac{1}{x-3}=-\infty$, $\lim_{x\to 3^+}\frac{1}{x-3}=+\infty$.

Squeeze theorem. If $\ell(x)\leq g(x)\leq u(x)$ near a and $\ell(x)\to L$, $u(x)\to L$, then $g(x)\to L$. Example: $3-(x-1)^2\leq g(x)\leq 3+(x-1)^2$ gives $\lim_{x\to 1}g(x)=3$. The bounds must agree at the point, and the lower bound must stay below the upper one.

More tools. $\lim_{x\to 0}\frac{\sin 5x}{2x}=\lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\frac{5}{2}=\frac{5}{2}$. Limit laws: if $f\to 8$ and $g\to -2$, then $\frac{f}{g}+3g\to -4-6=-10$ (never divide by a zero limit). If $\lim_{x\to 2}\frac{f(x)-7}{x-2}$ is finite, the top must go to 0, so $\lim_{x\to 2}f(x)=7$.

Where students lose points

1. Stopping at $\frac{0}{0}$ or writing "undefined": $\frac{0}{0}$ means do more algebra.
2. Mixing up $f(a)$ and $\lim_{x\to a}f(x)$.
3. Checking only one side: the limit exists only if both one-sided limits agree.
4. Dropping "lim" before you substitute.
5. Substituting before cancelling, and getting $\frac{0}{0}$ again.

Remember

- A limit is where $f(x)$ is heading as $x\to a$, not $f(a)$.
- $\lim_{x\to a}f(x)$ exists $\iff$ the left and right limits are equal.
- Try direct substitution first.
- $\frac{0}{0}\to$ factor and cancel, or multiply by the conjugate.

© 2026 S&D Academy

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

sad-academy.com

sad-academy.com

AP® is a registered trademark of College Board, which is not affiliated with and does not endorse this material.

 © 2026 S&D Academy