

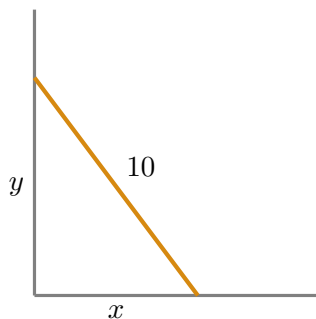
Related Rates & L'Hospital's Rule

AP Calculus AB | Unit 4 | related rates, linear approximation, L'Hospital

1. Related rates in four steps

The method

1. **Draw** it. Give a letter to every quantity that *changes*; constants stay numbers.
2. Write an **equation** that relates the letters.
3. **Differentiate** both sides with respect to t (every letter gets its own $\frac{d}{dt}$ factor).
4. **Only now** plug in the numbers and solve. Answer with units and direction.



The ladder. A 10 ft ladder; the bottom moves away at $\frac{dx}{dt} = 2$ ft/s. Find $\frac{dy}{dt}$ when $x = 6$.

$$x^2 + y^2 = 100 \implies 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.$$

When $x = 6$, $y = 8$: $2(6)(2) + 2(8)\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -1.5$ ft/s. The top is sliding *down* at 1.5 feet per second.

In general $\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$: as $y \rightarrow 0$ the top falls faster and faster.

The balloon. $\frac{dV}{dt} = 100$ cm³/s; find $\frac{dr}{dt}$ when $r = 5$.

$$V = \frac{4}{3}\pi r^3 \implies \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \implies 100 = 4\pi(25) \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{1}{\pi} \approx 0.318 \text{ cm/s}.$$

2. Linear approximation

Near $x = a$ the tangent line is almost the curve:

$$L(x) = f(a) + f'(a)(x - a).$$

$\sqrt{4.1}$: take $a = 4$, $f(4) = 2$, $f'(4) = \frac{1}{4}$. Then $\sqrt{4.1} \approx 2 + \frac{1}{4}(0.1) = 2.025$ (calculator: 2.02485).

Over or under?

Concave **down** near a : the tangent line lies above the curve, so it is an **overestimate**.

Concave **up** near a : the tangent line lies below the curve, so it is an **underestimate**.

3. L'Hospital's rule

Only for $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Differentiate the top and the bottom *separately*. This is not the quotient rule.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1, \quad \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

On the free response justify the rule first: write $\lim_{x \rightarrow 0} \sin x = 0$ and $\lim_{x \rightarrow 0} x = 0$ separately, then apply it. Never write $\frac{0}{0}$ as if it were a value.

Where students lose points

1. Plugging in numbers *before* differentiating.
2. Forgetting the chain-rule factor: $2x$ instead of $2x \frac{dx}{dt}$.
3. No units, or ignoring what the sign means.
4. L'Hospital's rule without showing the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
5. Using the quotient rule inside L'Hospital's rule.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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