Riemann Sums

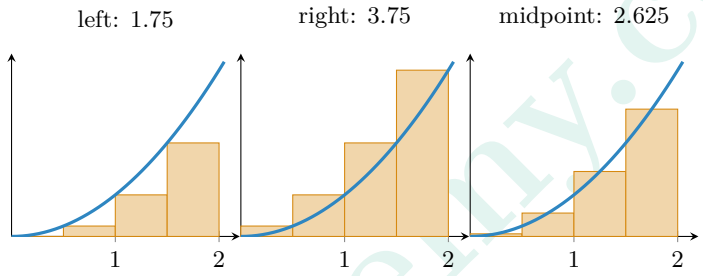
AP Calculus AB | Unit 6 | accumulation, left/right/midpoint/trapezoid, tables

1. Area is accumulated change

At a constant 30 m/s for 4 s, distance $= 30 \times 4 = 120$ m: height × width, the *area* under the velocity graph. In general, **the area under a rate graph is the accumulated change**, with units (rate units) × (input units): $\frac{\text{m}}{\text{s}} \times \text{s} = \text{m}$.

2. Left, right, midpoint, trapezoid

Cut $[a, b]$ into subintervals and add **width** × **height** for each. For $f(x) = x^2$ on $[0, 2]$ with $n = 4$ ($\Delta x = 0.5$):



- **Left**: $0.5\,[f(0) + f(0.5) + f(1) + f(1.5)] = 1.75$
- **Right**: $0.5\,[f(0.5) + f(1) + f(1.5) + f(2)] = 3.75$
- **Midpoint**: $0.5\,[f(0.25) + f(0.75) + f(1.25) + f(1.75)] = 2.625$
- **Trapezoidal**: the average of left and right, $\frac{1}{2}(1.75 + 3.75) = 2.75$

The exact area is $\frac{8}{3} \approx 2.667$.

3. Over or under?

No exact answer needed

f **increasing**: left sum is an underestimate, right sum is an overestimate.
f **decreasing**: left sum is an overestimate, right sum is an underestimate.
Trapezoidal sum: **concave up** ⇒ overestimate; **concave down** ⇒ underestimate.

Give the reason: *"the right Riemann sum is an overestimate because f is increasing."*

4. From a table, with unequal widths

t (s)	0	2	5	9
$v(t)$ (m/s)	20	32	44	52

Four values make **three** subintervals, of widths 2, 3 and 4. Multiply each width by its own height:

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- Left: $(2)(20) + (3)(32) + (4)(44) = 312$ m
- Right: $(2)(32) + (3)(44) + (4)(52) = 404$ m
- Trapezoidal: $\frac{1}{2}(312 + 404) = 358$ m

5. More rectangles: the definite integral

For x^2 on $[0, 2]$ the right sums are $3.750,\ 3.188,\ 2.922,\ 2.793,\ 2.729$ for $n = 4, 8, 16, 32, 64$: they close in on $\frac{8}{3}$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

6. Sums in sigma notation, and $F' = f$

Reading a limit of a sum. In $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{4i}{n}\right)^3 \frac{4}{n}$, the factor $\frac{4}{n}$ is Δx (an interval 4 wide) and $x_i = 1 + \frac{4i}{n}$ runs from 1 to 5 (right endpoints), so the limit is $\int_1^5 x^3\,dx = \frac{5^4 - 1^4}{4} = 156$ (equivalently $\int_0^4 (1+x)^3\,dx$).

More points. For increasing f a right sum overestimates; adding points makes the rectangles hug the curve and the overestimate shrinks.

Accumulation. $F(x) = \int_0^x f(t)\,dt$ has $F' = f$ and $F'' = f'$: F decreases where $f < 0$, increases where $f > 0$, and is concave up where f increases; a sign change of f from $-$ to $+$ gives a minimum of F.

Where students lose points

1. Using one Δx when the table has unequal widths.
2. Mixing up left and right, or using one rectangle too many.
3. Over/under with no reason.
4. Judging a trapezoidal sum by increasing/decreasing instead of concavity.
5. No units on an accumulation answer.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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