Separation of Variables

AP Calculus AB | Unit 7 | solving a differential equation exactly

1. Five steps

Example. $\dfrac{dy}{dx} = 2x\,(y+1)$ with $y(0) = 2$.

Separate, integrate (+ C now), initial condition, solve

1. **Separate:** every y with dy, every x with dx: $\dfrac{1}{y+1}\,dy = 2x\,dx$.
2. **Integrate both sides, with $+C$ in the same line:** $\ln|y+1| = x^2 + C$.
3. **Initial condition:** $\ln 3 = 0 + C$, so $C = \ln 3$.
4. **Solve for y:** $|y+1| = e^{x^2+\ln 3} = 3e^{x^2}$; $y+1 > 0$ at the start, so $y = 3e^{x^2} - 1$.
5. **Check:** $y(0) = 3 - 1 = 2$.

2. Where the $+C$ goes

Leaving out C and adding it at the end, $\ln|y+1| = x^2 \Rightarrow y+1 = e^{x^2} \Rightarrow y = e^{x^2} + C$, gives $y = e^{x^2} + 1$. It is **not** a solution: $\frac{dy}{dx} = 2x\,e^{x^2}$ but $2x(y+1) = 2x\left(e^{x^2}+2\right)$. A constant added on the logarithm side becomes a constant *factor* e^C after exponentiating.

Scoring. On the free response, no separation of variables earns no points for the part, and with no constant of integration the most you can earn is the first two points (the separation and the antiderivatives).

3. Choosing the sign

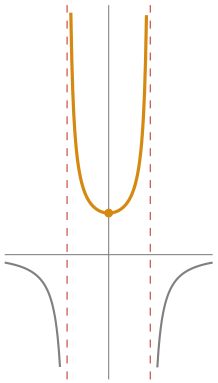
$\dfrac{dy}{dx} = \dfrac{4x}{y}$, $y(0) = -3$: $y\,dy = 4x\,dx$, $\dfrac{y^2}{2} = 2x^2 + C$, $C = \dfrac{9}{2}$, so $y^2 = 4x^2 + 9$. The solution must pass through $(0, -3)$, so $y = -\sqrt{4x^2+9}$: neither $\pm$ nor the positive root.

4. Where the solution lives

$\dfrac{dy}{dx} = 2xy^2$, $y(0) = 1$: $\dfrac{dy}{y^2} = 2x\,dx$, $-\dfrac{1}{y} = x^2 + C$, $C = -1$, so

$$y = \frac{1}{1-x^2}.$$

The formula blows up at $x = \pm 1$. A solution must be differentiable on an interval that contains the starting point, so this one lives on $-1 < x < 1$ only.



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Where students lose points

1. Integrating before every y is with dy and every x with dx.
2. No $+C$, or a $+C$ added after exponentiating.
3. Writing e^{x^2+C} as $e^{x^2}+C$: it is $e^C e^{x^2}$.
4. Ignoring the sign the initial condition demands for a square root or an absolute value.
5. Stopping at an equation in y when the question asks for y as a function of x.

Now practise it

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

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