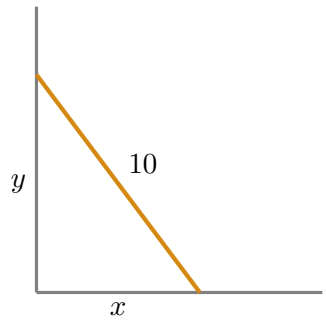
Related Rates

AP Calculus AB | Unit 4 | two quantities change, one equation ties them

1. The four steps

The method

1. **Draw** it. Give a letter to every quantity that *changes*; constants stay numbers. Write down the rate you know and the rate you want.
2. Write **one equation** that relates the letters. If it has a letter you have no rate for, eliminate that letter now (similar triangles, a given relationship).
3. **Differentiate** both sides with respect to t: every letter brings its own $\frac{d}{dt}$ factor (chain rule).
4. **Only now** plug in the numbers and solve. Answer with units and a direction.



The ladder. A 10 ft ladder; the bottom moves away at $\frac{dx}{dt} = 2$ ft/s. Find $\frac{dy}{dt}$ when $x = 6$.

$$x^2 + y^2 = 100 \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

When $x = 6$, $y = 8$: $2(6)(2) + 2(8)\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -1.5$ ft/s. The top is sliding *down* at 1.5 feet per second.

In general $\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$: as $y \to 0$ the top falls faster and faster.

2. The balloon

$\frac{dV}{dt} = 100$ cm^3/s; find $\frac{dr}{dt}$ when $r = 5$.

$$V = \tfrac{4}{3}\pi r^3 \implies \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \implies 100 = 4\pi(25)\frac{dr}{dt} \implies \frac{dr}{dt} = \frac{1}{\pi} \approx 0.318 \text{ cm/s}.$$

The radius grows *slower* as the balloon gets bigger: r^2 is in the denominator of $\frac{dr}{dt} = \frac{100}{4\pi r^2}$.

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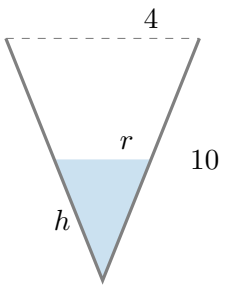
3. The cone: one letter too many



A cone-shaped tank, point down, 10 m deep with top radius 4 m. Water flows in at $\frac{dV}{dt} = 2$ m^3/min. Find $\frac{dh}{dt}$ when $h = 5$.

$V = \frac{1}{3}\pi r^2 h$ has three changing letters but only one known rate. The water is a cone of the same shape, so by **similar triangles** $\frac{r}{h} = \frac{4}{10}$, $r = \frac{2}{5}h$, and

$$V = \tfrac{1}{3}\pi\left(\tfrac{2}{5}h\right)^2 h = \tfrac{4\pi}{75}h^3,$$
$$\frac{dV}{dt} = \frac{4\pi}{25}h^2\frac{dh}{dt} \implies 2 = \frac{4\pi}{25}(5)^2\frac{dh}{dt} = 4\pi\frac{dh}{dt},$$
$$\frac{dh}{dt} = \frac{1}{2\pi} \approx 0.159 \text{ m/min}.$$

4. Two cars: the sign of a rate

Car B is 4 km east of a crossroads, driving away at 30 km/h: $x = 4$, $\frac{dx}{dt} = +30$. Car A is 3 km north, driving *toward* it at 60 km/h: $y = 3$, $\frac{dy}{dt} = -60$. The distance between them is z, with $z^2 = x^2 + y^2$ and $z = 5$ now.

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \implies 5\frac{dz}{dt} = 4(30) + 3(-60) = -60 \implies \frac{dz}{dt} = -12 \text{ km/h}.$$

The distance is *shrinking* at 12 km/h. A quantity that decreases has a negative rate; with +60 for car A the answer would have the wrong sign and the wrong size.

5. Angles, products, shadows, and chains

An angle. A drone flies 40 m up, moving away at 5 m/s. With $\tan\theta = \frac{40}{x}$: $\sec^2\theta\,\frac{d\theta}{dt} = -\frac{40}{x^2}\frac{dx}{dt}$. At $x = 30$ the line of sight is 50, $\sec^2\theta = \frac{25}{9}$, so $\frac{d\theta}{dt} = -0.08$ rad/s.

Two changing factors. $A = LW$ gives $\frac{dA}{dt} = L\frac{dW}{dt} + W\frac{dL}{dt}$; with $L = 10$, $L' = 3$, $W = 6$, $W' = -2$: $10(-2) + 6(3) = -2$ cm^2/s.

A shadow. Light 18 ft up, a person 6 ft tall walking away at 4 ft/s: $\frac{18}{x+s} = \frac{6}{s}$ gives $s = \frac{1}{2}x$, so $\frac{ds}{dt} = 2$ ft/s wherever the person is; the tip of the shadow moves at $4 + 2 = 6$ ft/s.

A chain of rates. $\frac{dT}{dt} = \frac{dT}{da}\cdot\frac{da}{dt} = (-0.0065)(4) = -0.026$ °C per second. If a sphere's radius grows at a constant rate, $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$ grows without bound; at a fixed r, doubling $\frac{dr}{dt}$ doubles $\frac{dV}{dt}$.

Where students lose points

1. Plugging in numbers *before* differentiating (a constant's derivative is 0).
2. Forgetting a rate: $2x$ instead of $2x\,\frac{dx}{dt}$.
3. Differentiating while the equation still has a letter you have no rate for.
4. The wrong sign: anything that shrinks has a negative rate.
5. No units, or no sentence saying what the answer means.

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