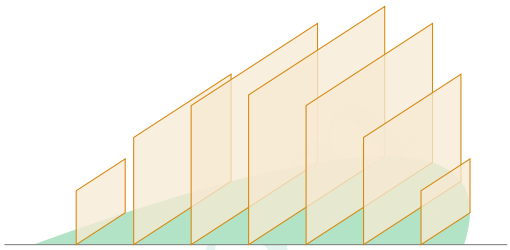
# Volumes with Known Cross Sections

AP Calculus AB | Unit 8 | slice it, add the slices

## 1. The idea

A solid stands on a base region. Every slice perpendicular to the $x$-axis is a known shape whose size depends on $x$. A thin slice has volume $A(x)\,\Delta x$; adding the slices gives



**Volume by slicing**

$$V = \int_a^b A(x)\,dx,$$

where $A(x)$ is the area of the cross section at $x$, and its side $s$ is the width of the base there.

## 2. Squares and other shapes

**Base:** the region under $y = 9 - x^2$, above the $x$-axis, for $-3 \le x \le 3$. The side of each slice is top − bottom: $s(x) = 9 - x^2$.

| cross section | area $A(x)$ | volume |
|---|---|---|
| square | $s^2$ | $\int_{-3}^{3} (9-x^2)^2\,dx = \frac{1296}{5} \approx 259.2$ |
| semicircle, diameter $s$ | $\frac{\pi}{8}s^2$ | $\frac{162\pi}{5} \approx 101.8$ |
| equilateral triangle | $\frac{\sqrt{3}}{4}s^2$ | $\frac{324\sqrt{3}}{5} \approx 112.2$ |
| isosceles right triangle, leg $s$ | $\frac{1}{2}s^2$ | $\frac{648}{5} = 129.6$ |
| rectangle of height 2 | $2s$ | $2\int_{-3}^{3} (9-x^2)\,dx = 72$ |

Only the area formula changes. There is no $\pi$ unless the cross section is round: nothing rotates in these problems.

## 3. Slices perpendicular to the $y$-axis

**Base:** the triangle bounded by $y = 2x$, the $x$-axis and $x = 2$. Slices perpendicular to the $y$-axis are horizontal, from the left boundary $x = \frac{y}{2}$ to the right boundary $x = 2$:

$$s(y) = 2 - \frac{y}{2}, \quad 0 \le y \le 4, \qquad V_{\text{squares}} = \int_0^4 \left(2 - \frac{y}{2}\right)^2 dy = \frac{16}{3}.$$

Write both boundaries as $x$ in terms of $y$, and use $y$-limits.

sad-academy.com

 © 2026 S&D Academy

**Where students lose points**

1. Integrating the side $s$ instead of the area $A(x)$ (that gives the area of the base).
2. Semicircles: using $s$ as the radius instead of $\frac{s}{2}$.
3. Adding a $\pi$ that does not belong.
4. Slices perpendicular to the $y$-axis, but the side and the limits still written in $x$.
5. Limits that cover only part of the base.

**Now practise it**

Theory sticks once you use it. On **sad-academy.com** you'll find practice questions for this exact subtopic in the AP format, and an AI that grades your free-response answers against the scoring guidelines, the way an AP reader would.

**sad-academy.com**

sad-academy.com

AP® is a registered trademark of College Board, which is not affiliated with and does not endorse this material.

 © 2026 S&D Academy